# SM3 9.2 Pythagorean Identity

A right triangle is constructed with a hypotenuse of length $r$ and an acute angle $\theta$.

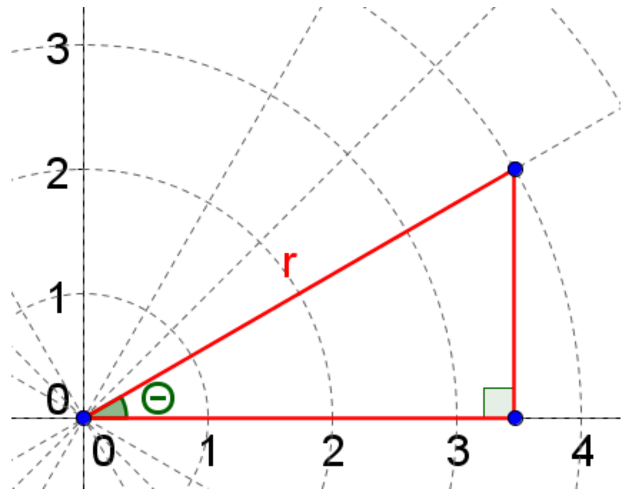
In terms of $r$ and $\theta$, how long are the legs of the triangle?

Use the trigonometric definitions to solve for $opp$ and $adj$ in terms of $r$ and $\theta$:

$$\sin(\theta) = \frac{opp}{hyp} \qquad \cos(\theta) = \frac{adj}{hyp}$$

$$\sin(\theta) = \frac{opp}{\phantom{hyp}} \qquad \cos(\theta) = \frac{adj}{\phantom{hyp}}$$

$$= opp \qquad = adj$$




Add the values of the lengths of the legs in terms of $r$ and $\theta$ to the picture.

The Pythagorean Identity is derived from the Pythagorean Theorem, $a^2 + b^2 = c^2$, by substituting the lengths of the triangle in the picture into the Pythagorean Theorem.

Follow the steps of the proof to derive the Pythagorean Identity (the last statement of the proof, directly above QED, is the Pythagorean Identity.

| **Statements** | **Reasons** |
|---|---|
| $a^2 + b^2 = c^2$ | Given |
| $(\qquad)^2 + (\qquad)^2 = r^2$ | Substitution |
| $\qquad + \qquad = r^2$ | Multiplication |
| $\qquad + \qquad = 1$ | Divide by $r^2$ |
| QED | |

Notice that because the Pythagorean Identity does not contain $r$, it works for any $r$.

Let's make the Pythagorean Identity more malleable by manipulating its terms. Complete each proof by using the Pythagorean Identity as the given and following the reasons.

| | |
|---|---|
| $\qquad + \qquad = 1$ | Given |
| | Subtract $\cos^2\theta$ |
| QED | |

| | |
|---|---|
| $\qquad + \qquad = 1$ | Given |
| | Subtract $\sin^2\theta$ |
| QED | |

We now have 3 equations that are each essentially the Pythagorean Identity. However, all of them only include $\sin\theta$ and $\cos\theta$. Let's develop some alternate versions of that include other trig functions.

Complete each exploration by using the Pythagorean Identity as the given and following the reasons.

| | |
|---|---|
| $\sin^2\theta + \cos^2\theta = 1$ | Given |
| $\frac{\sin^2\theta}{\_\_\_\_} + \frac{\cos^2\theta}{\_\_\_\_} = \frac{1}{\_\_\_\_}$ | Divide by $\cos^2\theta$ |
| | Definition of tan,sec |
| QED | |

| | |
|---|---|
| $\sin^2\theta + \cos^2\theta = 1$ | Given |
| $\frac{\sin^2\theta}{\_\_\_\_} + \frac{\cos^2\theta}{\_\_\_\_} = \frac{1}{\_\_\_\_}$ | Divide by $\sin^2\theta$ |
| | Definition of cot,csc |
| QED | |

The above identities can also move one term to the other side, giving us even more identities!

Write the conclusions of the above explorations in two other forms each by subtracting a term from the right to the left side of the equation.

| | |
|---|---|
| $\tan^2\theta + 1 = \sec^2\theta$ | Given |
| | Subtract 1 |
| QED | |

| | |
|---|---|
| $1 + \cot^2\theta = \csc^2\theta$ | Given |
| | Subtract 1 |
| QED | |

| | |
|---|---|
| $\tan^2\theta + 1 = \sec^2\theta$ | Given |
| | Subtract $\tan^2\theta$ |
| QED | |

| | |
|---|---|
| $1 + \cot^2\theta = \csc^2\theta$ | Given |
| | Subtract $\cot^2\theta$ |
| QED | |

# We're up to 9 equations that are all essentially the Pythagorean Identity!

I know, as honorable math students, you're eager to start using square roots, conjugates, and other ideas to come up with dozens more. We could make hundreds of identities! We could fill the world's libraries with all sorts of unique, innovations by manipulating the Pythagorean Identity.

Sadly, we're not going to spend the rest of your lives producing more Pythagorean Identities. We're just going to memorize these 9 equations and make use of them. You may exhale a sigh of (faux) regret.

| | | |
|---|---|---|
| *Memorize*: | $\sin^2\theta + \cos^2\theta = 1$ | The Pythagorean Identity |
| | $\sin^2\theta = 1 - \cos^2\theta$ | The Pythagorean Identity, solved for $\sin^2\theta$ |
| | $\cos^2\theta = 1 - \sin^2\theta$ | The Pythagorean Identity, solved for $\cos^2\theta$ |
| | $\tan^2\theta + 1 = \sec^2\theta$ | The Pythagorean Identity, divided by $\cos^2\theta$ |
| | $\tan^2\theta = \sec^2\theta - 1$ | The Pythagorean Identity, divided by $\cos^2\theta$, then $-1$. |
| | $1 = \sec^2\theta - \tan^2\theta$ | The Pythagorean Identity, divided by $\cos^2\theta$, then $-\tan^2\theta$. |
| | $1 + \cot^2\theta = \csc^2\theta$ | The Pythagorean Identity, divided by $\sin^2\theta$ |
| | $\cot^2\theta = \csc^2\theta - 1$ | The Pythagorean Identity, divided by $\sin^2\theta$, then $-1$. |
| | $1 = \csc^2\theta - \cot^2\theta$ | The Pythagorean Identity, divided by $\sin^2\theta$, then $-\cot^2\theta$. |

You may represent any of the above substitutions in a proof as Pyth ID.

You may use the Pyth IDs in either direction (e.g., you may replace $\sin^2\theta + \cos^2\theta$ with the number 1 or you may change the number 1 into $\sin^2\theta + \cos^2\theta$).

*Example*: Prove $\sin^3\theta\cos^4\theta = \cos^4\theta\sin\theta - \cos^6\theta\sin\theta$

Strategy: The right side has two terms, so we just need replace a portion of the left side by exchanging a term for two terms. We want less $\sin\theta$ power on the right side, so let's replace $\sin^2\theta$ on the left using a Pyth ID.

I'm only going to write the left side after the first line to save space.

| **Statements** | | **Reasons** |
|---|---|---|
| $\sin^3\theta\cos^4\theta =$ | $\cos^4\theta\sin\theta - \cos^6\theta\sin\theta$ | Given |
| $\sin^2\theta\cos^4\theta\sin\theta =$ | | Factor |
| $(1-\cos^2\theta)\cos^4\theta\sin\theta =$ | | Pyth ID |
| $\cos^4\theta\sin\theta - \cos^6\theta\sin\theta =$ | | Distribution |

QED

One of the most important aspects of this skill will become clear next year during your study of integral calculus. Trigonometric identities allow you to alter expressions to be more suited to easier manipulation with calculus techniques.

HW9.2 S

Use two columns to prove each identity. You may use the equation as the first statement.

1)

| Statements | Reasons |
| --- | --- |
| $2\cos^2\theta - 1 = 1 - 2\sin^2\theta$ | Given |

2)

| Statements | Reasons |
| --- | --- |
| $\tan^2\theta = \dfrac{1-\cos^2\theta}{\cos^2\theta}$ | Given |

3) $4\sin^2\theta + 4\cos^2\theta = 4$

4) $\cos\theta - \cos^3\theta = \cos\theta\sin^2\theta$

5) $\dfrac{\cos^2\theta - 1}{\cos\theta} = -\tan\theta\sin\theta$

6) $\dfrac{\sec\theta + 1}{\tan\theta} = \dfrac{\sin\theta}{1-\cos\theta}$

7) $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$

8) $\tan^4\theta + \tan^2\theta = \sec^4\theta - \sec^2\theta$

9) $(1-\tan\theta)^2 = \sec^2\theta - 2\tan\theta$

10) $(\cos\theta - \sin\theta)^2 = 1 - 2\sin\theta\cos\theta$

11) $$\frac{\cos^2\theta}{1-\sin\ \theta} = 1 + \sin\theta$$

12) $$(\sec^2\theta + \csc^2\theta) - (\tan^2\theta + \cot^2\theta) = 2$$

13) $$\frac{1}{1-\cos\ \theta} + \frac{1}{1+\cos\ \theta} = \frac{2}{\sin^2\theta}$$

14) $$\frac{\sec^2\theta\csc\ \theta}{\sec^2\theta + \csc^2\theta} = \sin\theta$$